Design of Unknown Input Reconstruction Algorithm in Presence of Measurement Noise

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Abstract: In this paper a novel approach to unknown input estimation based on a parity equations concept is developed. Unlike the unknown input observers based on the Kalman filtering approach, the observer proposed here is independent of the system state vector. Therefore, due to the reduction of the number of estimated signals, a higher accuracy of the input estimation is achieved. This makes the scheme advantageous in cases when the accuracy of the input estimate is crucial and the knowledge about the system states is not required. By increasing the order of the parity space, which is a tuning parameter of the algorithm proposed, the new approach allows the influence of the effects of measurement noise to be reduced. A Lagrange multiplier method is used to obtain an analytical solution for the filter parameters.

Keywords: Filtering, observers, parity equations, unknown input reconstruction

1. INTRODUCTION

The history of observers dates back to the 1960s with the Luenberger system state observers (see Luenberger (1964)). Subsequently, state observers have been extended to the class of systems with both, known and unknown system inputs (see for example Darouach and Zasadzinski (1997)). Over the last decade the simultaneous estimation of both, state vector and unknown inputs, based on a Kalman filtering approach has gained an interest (cf. Floquet and Barbot (2006); Hsieh (2000)). Gillijns and De Moor (2007a) combined the state observer proposed by Darouach and Zasadzinski (1997) and the unknown input estimator of Hsieh (2000) creating a state and unknown input observer, which is optimal in the minimum variance sense. This approach has subsequently been extended to the case of a linear system with a direct feedthrough (see Gillijns and De Moor (2007b)).

In this paper a new approach to the problem of unknown input estimation based on parity equations (PE) is proposed. A detailed explanation of PE can be found in Ding (2008); Gertler (1991); Li and Shah (2002). A very general relationship between the PE and the left inverse of the minimum-phase deterministic system has been presented by Edelmayer (2005). On the contrary, in this paper PE are used to obtain an approximation (estimate) of the unknown input of a stochastic system, whose measurements are affected by noise. The method is suitable for both minimun-phase and nonminimum-phase systems. The contribution of this paper is to utilise the Lagrange multiplier method to provide an analytical solution for the filter parameters, which minimise effects of measurement noise. Furthermore, unlike the unknown input observers (UIOs) based on the Kalman filtering approach (see for example Gillijns and De Moor (2007a,b)), the developed observer is orthogonal to the system state vector.

In the framework of this paper, firstly, the PE theory is explained. Subsequently, the derivation of the novel filter is provided. Use is made of the Lagrange multiplier method (see for example Bertsekas (1982)) to obtain an analytical solution to the unknown input estimator parameters. Then, based on a numerical example, the influence of the parity space order (which is a tuning parameter of the filter) on the efficacy of the algorithm is analysed. Finally, the accuracy of the novel method is compared to that of the Kalman filter-based minimum variance unbiased unknown input estimator proposed by Gillijns and De Moor (2007b).

2. DESCRIPTION OF APPROACH

In this section the new algorithm is derived. Firstly, for completeness, PE are described in Subsection 2.1, see e.g. Li and Shah (2002). Then, in Subsection 2.2, using existing concepts, a new unknown input observer based on PE (further referred to as PE-UIO) is developed.

2.1 Parity Equations

Assume that a linear dynamic discrete time two-input single-output system is represented by an n^{th} order state space equation of the following form:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu_0(t) + Gv(t) \\ y_0(t) &= Cx(t) + Du_0(t) + Hv(t) \\ u(t) &= u_0(t) + \tilde{u}(t) \\ y(t) &= y_0(t) + \tilde{y}(t) \end{aligned} \tag{1}$$

where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times 1}$, $C \in \mathcal{R}^{1\times n}$, $D \in \mathcal{R}^{1\times 1}$, $G \in \mathcal{R}^{n\times 1}$ and $H \in \mathcal{R}^{1\times 1}$. The terms $u_0(t)$, $v(t)$ and $y_0(t)$ refer to, respectively, known and unknown input to the system and the system output. An errors-in-variables (EIV) case is considered (see, for example, Söderström (2007)), i.e. all measured variables, which are input $u(t)$ and $y(t)$, are affected by a zero mean, white Gaussian mutually uncorrelated measurement noise sequences denoted by $\tilde{u}(t)$ and $\tilde{y}(t)$, respectively. Hence, the noise free but unmeasured system input and output are denoted as $u_0(t)$ and $y_0(t)$, respectively.

The following stacked vector of the unknown input, $v(t)$, is created (see, for example, Li and Shah (2002)):

$$V = [\, v(t-s)\ v(t-s+1)\ \cdots\ v(t)\,]^T \tag{2}$$

where the term s denotes the order of the parity space. Analogously, one can build stacked vectors of $y(t)$, $y_0(t)$, $\tilde{y}(t)$, $u(t)$, $u_0(t)$ and $\tilde{u}(t)$ which are denoted, respectively, as Y, Y_0, $\tilde{Y}$, U, U_0 and $\tilde{U}$. By making use of this notation the system (1) can be expressed in the form of:

$$Y_0 = \Gamma x(t-s) + QU_0 + TV \tag{3}$$

where Γ is an extended observability matrix:

$$\Gamma = \left[\, C^T\ A^TC^T\ \cdots\ (A^s)^TC^T \,\right]^T \in \mathcal{R}^{(s+1)\times n} \tag{4}$$

and Q is the following block Toeplitz matrix:

$$Q = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{s-1}B & CA^{s-2}B & \cdots & D \end{bmatrix} \in \mathcal{R}^{(s+1)\times(s+1)} \tag{5}$$

Analogously, the matrix T is built by replacing D with H and B with G in the matrix Q. In order to eliminate the unknown state vector from (3), a row vector W is defined, which belongs to the left nullspace of Γ, i.e.

$$W\Gamma = 0 \tag{6}$$

Hence (3) can be reformulated as:

$$WTV = WY_0 - WQU_0 = W(Y - \tilde{Y}) - WQ(U - \tilde{U}) \tag{7}$$

By rearranging the measured (known) variables on the right-hand side of (7) and the unknowns on the left-hand side, the following PE is obtained (cf. Li and Shah (2002)):

$$WTV + W\tilde{Y} - WQ\tilde{U} = WY - WQU \tag{8}$$

In the next section use is made of the PE in order to derive a novel algorithm for the unknown input estimation.

2.2 Input reconstruction with measurement noise filtering

Denote the matrix spanning the left nullspace of Γ as $\Gamma^{\perp}$. Consequently, the row vector W is a linear combination of rows of $\Gamma^{\perp}$. It is assumed here that the system (1) is observable, hence the extended observability matrix Γ is of full rank. Therefore, the dimension of $\Gamma^{\perp}$ is $(s - n + 1) \times (s + 1)$, and since T is square, it is true that $\Gamma^{\perp}T \in \mathcal{R}^{(s-n+1)\times(s+1)}$. Thus in the case of noise-free input and output measurements, i.e. when $U{=}U_0$ and $Y{=}Y_0$, the following equation holds (cf. (7)):

$$\Gamma^{\perp}TV = b \tag{9}$$

where b is a column vector of $(s - n + 1)$ elements, and:

$$b = \Gamma^{\perp}Y - \Gamma^{\perp}QU \tag{10}$$

Note, that the matrix T consists of the Markov parameters of the relation between the unknown input and the output, which are given by (see Kirtikar et al. (2009)):

$$T_i = \begin{cases} H & ,i = 0 \\ CA^{i-1}G & ,i > 0 \end{cases} \tag{11}$$

The relative degree of the system $G_v(z) = C(zI - A)^{-1}G + H$, denoted as r, is the smallest number for which $T_r \neq 0$ (cf. Edelmayer (2005)). Hence, one can note that (10) is a homogenous set of equations (i.e. the sequence of unknown input values can be determined explicitly from (10)) only, if the system $G_v(z)$ has no zeros, i.e. the relative degree is equal to the order of $G_v(z)$. (Which means that the last r columns of the matrix $\Gamma^{\perp}T$ are equal zero.) Nevertheless, the unique solution to the set of equations (10) can be seriously affected by the measurement noise $\tilde{u}(t)$ and $\tilde{y}(t)$. The algorithm proposed here minimises the effects of the unwanted measurement noise. Furthermore, the technique can be utilised to yield an approximation of $v(t)$ in the case when $G_v(z)$ has zeros. The proposed method is suitable for both minimum-phase and nonminimum-phase systems.

It is proposed to calculate the value of the unknown input as:

$$\hat{v}(t) = WY - WQU \tag{12}$$

which, in the case of a noise-free input and output measurements, is:

$$\hat{v}(t) = WTV \tag{13}$$

Thus, based on the assumption that the unknown input is slowly varying, its estimate can be calculated as a linear combination of the sequence $v(t-s)$, $v(t-s+1), \cdots, v(t)$.

$$\hat{v}(t) = \alpha_0 v(t) + \alpha_1 v(t-1) + \cdots + \alpha_s v(t-s) \tag{14}$$

where the α parameters are dependent on the choice of the vector W, such that:

$$WT = [\, \alpha_s\ \alpha_{s-1}\ \cdots\ \alpha_0 \,]^T \tag{15}$$

One can note that (14) is an equation of a moving average finite impulse response filter with the gain given by the sum of the α parameters, i.e. the sum of elements of the vector WT. Thus, it is suggested here that the vector W should be selected in such a way, that the sum of elements of the vector WT is equal unity.

It is anticipated that the choice of the order of the parity space s, as well as the vector W may influence a lag in the estimate of the unknown input (due to the moving average filtering property of the unknown input estimator).

In the next subsection an algorithm for the selection of an optimal vector W is derived based on the Lagrange multiplier method.

2.3 Selection of optimal W

In the case of noisy input and output measurements, equation (12) becomes:

$$\hat{v}(t) = WTV + W\tilde{Y} - WQ\tilde{U} \tag{16}$$

Hence, the estimate of the unknown input is affected by a coloured noise. However, by a careful choice of W, the degrading effect of noise can be minimised. Due to the fact that $\tilde{y}(t)$ and $\tilde{u}(t)$ are uncorrelated, white and zero mean (i.e. the expected values $E\{\tilde{y}(t)\} = E\{\tilde{u}(t)\} = 0$), it is true that:

$$E\{W\tilde{Y} - WQ\tilde{U}\} = 0 \tag{17}$$

Hence asymptotically, a presence of the measurement noise does not cause a bias in the unknown input estimate. Furthermore, an influence of the measurement noise on the unknown input estimate can be minimised by reducing the variance of the term $W\tilde{Y} - WQ\tilde{U}$, i.e.:

$$E\{(W\tilde{Y} - WQ\tilde{U})(W\tilde{Y} - WQ\tilde{U})^T\} = W\Sigma_{\tilde{y}}W^T + \\ + WQ\Sigma_{\tilde{u}}Q^TW^T - W\Sigma_{\tilde{u}\tilde{y}}^TQ^TW^T - WQ\Sigma_{\tilde{u}\tilde{y}}W^T \quad (18)$$

where $\Sigma_{\tilde{u}} = E\{\tilde{U}\tilde{U}^T\}$, $\Sigma_{\tilde{y}} = E\{\tilde{Y}\tilde{Y}^T\}$, $\Sigma_{\tilde{u}\tilde{y}} = E\{\tilde{U}\tilde{Y}^T\}$. Due to the fact that the input and output measurement sequences are considered to be white, zero mean and mutually uncorrelated:

$$\Sigma_{\tilde{u}} = \sigma_{\tilde{u}}^2 I, \quad \Sigma_{\tilde{y}} = \sigma_{\tilde{y}}^2 I, \quad \Sigma_{\tilde{u}\tilde{y}} = 0 \quad (19)$$

where the terms $\sigma_{\tilde{u}}^2$ and $\sigma_{\tilde{y}}^2$ refer to the variance of the measurement error of the system input and output, respectively, whilst I is an identity matrix of appropriate dimension.

Subsequently, the vector W should be selected to minimise the cost function $f(W)$:

$$f(W) = W\Sigma_{\tilde{y}}W^T + WQ\Sigma_{\tilde{u}}Q^TW^T \quad (20)$$

subject to the following constraints:

(1) Sum of elements of WT is equal to 1.
(2) $W\Gamma = 0$.

The cost function (20) can be minimised by making use of the Lagrange multipliers method (see, for example, Bertsekas (1982)). Denote the rows of $\Gamma^\perp$ by γ_1, γ_2, ..., $\gamma_{(s-n+1)}$:

$$\Gamma^\perp = \left[\, \gamma_1^T \;\gamma_2^T \;\cdots\; \gamma_{(s-n+1)}^T \,\right]^T \quad (21)$$

The vector W is a linear combination of rows of $\Gamma^\perp$, i.e.

$$W = \sum_{i=1}^{s-n+1} p_i\gamma_i \quad (22)$$

Hence the cost function (20) can be reformulated as a function of the parameter vector $P = [\, p_1 \; p_2 \; \cdots \; p_{s-n+1} \,]^T$:

$$f(P) = \left(\sum_{i=1}^{k} p_i\gamma_i\right)\Sigma\left(\sum_{j=1}^{k} p_j\gamma_j^T\right) = \sum_{i=1}^{k}\sum_{j=1}^{k} p_ip_j\gamma_i\Sigma\gamma_j^T \quad (23)$$

where $k = s - n + 1$ and:

$$\Sigma = \Sigma_{\tilde{y}} + Q\Sigma_{\tilde{u}}Q^T \quad (24)$$

The cost function $f(P)$ is required to be minimised subject to the constraint:

$$g(P) = \mathrm{sum}_{row}(WT) - 1 = 0 \quad (25)$$

where the operator $\mathrm{sum}_{row}(A)$ denotes a column vector whose elements are sums of the appropriate rows of the matrix A.

The solution to the Lagrange minimisation problem is given by (see Bertsekas (1982)):

$$\nabla f(P) = \lambda\nabla g(P) \quad (26)$$

The partial derivative of $f(P)$ with respect to the i^{th} element of the vector P (denoted as p_i) is:

$$\frac{\partial f(P)}{\partial p_i} = \sum_{j=1}^{k} p_j\gamma_i\Sigma\gamma_j^T + \sum_{j=1}^{k} p_j\gamma_j\Sigma\gamma_i^T \quad (27)$$

After some manipulations the gradient of $f(P)$ is reformulated as:

$$(\nabla f(P))^T = \left(\Gamma^\perp\Sigma(\Gamma^\perp)^T + (\Gamma^\perp\Sigma\left(\Gamma^\perp)^T\right)^T\right)P \quad (28)$$

The partial derivative of the constraint function $g(P)$ with respect to p_i is calculated via:

$$\frac{\partial g(P)}{\partial p_i} = \mathrm{sum}_{row}(\gamma_iT) \quad (29)$$

Thus, the gradient of $g(P)$ can be reformulated as:

$$(\nabla g(P))^T = \mathrm{sum}_{row}(\Gamma^\perp T) \quad (30)$$

By making use of the notation:

$$S = \left(\Gamma^\perp\Sigma(\Gamma^\perp)^T + (\Gamma^\perp\Sigma\left(\Gamma^\perp)^T\right)^T\right) \quad (31)$$

and

$$\psi = \mathrm{sum}_{row}(\Gamma^\perp T) \quad (32)$$

the solution to the Lagrange optimisation problem (26) can be rewritten as:

$$SP = \lambda\psi \quad (33)$$

Hence, the optimal parameter vector P is given by:

$$P = \lambda S^{-1}\psi \quad (34)$$

The constraint function $g(P) = 0$ can be rewritten as:

$$P^T\psi - 1 = 0 \quad (35)$$

Incorporating (34) into (35):

$$\lambda\left(S^{-1}\psi\right)^T\psi - 1 = 0 \quad (36)$$

Hence, the Lagrange multiplier is given by:

$$\lambda = \left(\left(S^{-1}\psi\right)^T\psi\right)^{-1} \quad (37)$$

The algorithm for calculating the optimal vector W is summarised below:

(1) Select the order of the parity space $s \geq n$ and build matrices Γ, Q and T.
(2) Obtain $\Gamma^\perp$ (the left nullspace of Γ).
(3) Compute Σ using (24).
(4) Calculate the column vector ψ and the matrix S making use of (32) and (31), respectively.
(5) Obtain the Lagrange multiplier λ using (37).
(6) Calculate the parameter vector P by (34).
(7) Compute the vector W as:

$$W = P^T\Gamma^\perp \quad (38)$$

3. NUMERICAL EXAMPLE

Consider an exemplary system, described by (1), whose state space matrices are:

$$A = \begin{bmatrix} 0 & 0.765 \\ 1 & -0.050 \end{bmatrix} B = \begin{bmatrix} 0.005 \\ 0.5 \end{bmatrix} \quad G = \begin{bmatrix} 1.383 \\ 0.975 \end{bmatrix} \\ C = [\, 0 \;\; 2 \,] \qquad D = [0] \qquad H = [1] \quad (39)$$

The efficacy of the PE-UIO filter, designed for the system (39), for different cases of s and different cases of the input and output measurement noise variances is evaluated. Two efficiency indices are considered in order to assess the efficacy of the algorithms examined, namely the minimal mean square error (MSE_{min}) and the estimation lag (EL). Consider a classical mean square error (MSE) index, where the true input $v(t)$ is delayed by i samples with respect to the estimated input $\hat{v}(t)$:

$$\text{MSE}(i) = \frac{\sum_t \left(\hat{v}(t) - v(t-i)\right)^2}{\sum_t v^2(t-i)} \quad (40)$$

The word *minimum* here refers to the the fact that the minimal value of MSE, as a function of the delay i, is considered (i.e. MSE_{min}). The delay, for which the function MSE(i) achieves its minimum, is denoted EL (EL is the argument of MSE(i), i.e. i =EL):

$$\begin{aligned} \text{EL} &= \arg\min_i \text{MSE}(i) \\ \text{MSE}_{min} &= \text{MSE(EL)} \end{aligned} \quad (41)$$

Hence, the EL indicates the number of samples by which the unknown input estimate is delayed with respect to the input. The MSE_{min} provides the accuracy measure of the unknown input estimate (delayed by the EL).

A Monte-Carlo simulation comprising of 100 runs has been carried out. Mean values of the MSE_{min} and the EL for each simulation setup are presented in Table 1. As expected, an increase in the parity space order results in a corresponding increase in the EL. For the first two cases of the measurement noise $(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}})$, i.e. ((0.1,2) and (1,1)) the MSE_{min} reduces as s increases from 3 to 6, however, a further increase of s degrades the efficacy of the PE-UIO (in terms of MSE_{min}).

The last row of Table 1 shows the efficacy of the Kalman filter-based minimum variance unbiased (MVU) state and input estimator proposed by Gillijns and De Moor (2007b). Due to the moving average filtering properties of the PE-UIO and that only a single signal (the unknown input) is estimated, the PE-UIO appears to be advantageous in comparison to the MVU approach in the case examined. Fig. 1 presents an exemplary visual illustration of the unknown input estimation using MVU and PE-UIO with $s = 6$.

Table 1. Simulation results $(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}})$

$(\sigma^2_{\tilde{u}}, \sigma^2_{\tilde{y}})$	(0.1,2)		(1,1)		(10,1)	
s	EL	MSE_{min}	EL	MSE_{min}	EL	MSE_{min}
3	0	2.454	0	2.764	0	13.639
4	0	2.268	0	2.210	0	6.469
6	1	1.572	1	1.422	1	3.012
8	2	2.496	2	2.086	2	2.668
MVU	0	19.4971	0	18.841	0	28.7843

4. CONCLUSIONS AND FURTHER WORK

A new approach for reconstructing the unknown input has been developed. The main advantage of the new scheme is that it does not rely on state estimation. Therefore, since the number of estimated signals is reduced, the estimation accuracy (in terms of the discrepancy between the true and the estimated input) is increased. Consequently, the proposed approach offers advantages in cases, when knowledge about the system states is not required. Due to the moving average filtering properties of the proposed scheme, the effect of measurement noise on both signals, i.e input and output, is minimised.

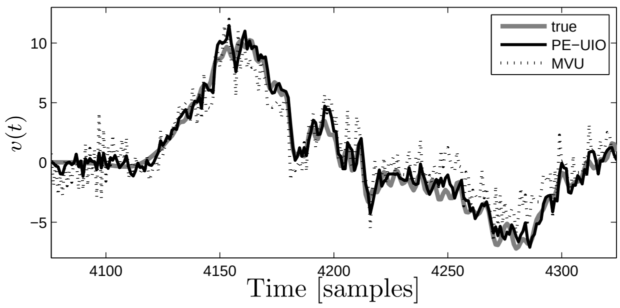

Fig. 1. Comparison of efficacy of PE-UIO and MVU ($\sigma^2_{\tilde{u}} = 1$, $\sigma^2_{\tilde{y}} = 1$)

Further work aims towards the development of the algorithm for multiple-input multiple-output systems. Consideration is also to be given to develop the algorithm for the more practical case of coloured measurement noise. Furthermore, the problem of choice of the order of the parity space depending on the noise variance remains open.

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